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Triton binding energy by the Faddeev approach with two-body local interactions

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Abstract. An analytical and separable representation for the two-particle S wave t matrix, for the central Yukawa interactions, has been suggested, which is separable and symmetric in momentum variables p and p'. The sturmian eigenfunctions for the S wave Hulthén potential, which form a complete set, have been used as the expansion bases. For the neutron–proton system, the triplet and the singlet states have been considered, including the soft-core repulsion in the singlet interaction. The resulting t matrix has been used as an input in the Faddeev equation for the neutron–deuteron collision problem, the triton binding energy has been computed and compared with the theoretical calculation of Malfliet and Tjon as well as the experimental findings.

1. Introduction

Several workers have used the formalism of Faddeev (1961) to calculate the triton binding energy of the three-nucleon systems, with nonlocal separable potentials, which are, however, somewhat unrealistic. Malfliet and Tjon (1969) have calculated the triton binding energy in the framework of the Faddeev equations with local Yukawa interactions by solving numerically the coupled two dimensional integral equations. They have also included the soft-core repulsion in the singlet state. The off-shell t matrices for such interactions were also obtained by them through the numerical solution of the Lippmann–Schwinger equation. Kharchenko and Petrov (1969) have investigated the neutron–deuteron collision problem with the local Hulthén and square-well potentials, using the corresponding sturmian functions as expansion bases. Purkayastha *et al* (1972) have also applied the Faddeev equation to the trinucleon problem using the local Hulthén potential but with the deuteron wavefunctions as expansion bases. However, it is well known that the interaction between two nucleons is well represented by a sum of two or more Yukawa potentials, thereby enabling the resulting local potential to contain a repulsive core and an attractive outer region (Coester and Yen 1963, Mongan 1969). Hence it will be interesting to investigate the three-nucleon problem with two-body local Yukawa potentials. But, the sturmian eigenfunctions corresponding to the Yukawa potential are not known analytically and as such the two-body off-shell partial-wave t matrix is to be obtained numerically. This will cause considerable mathematical difficulty in the application of the Faddeev formalism to the three-body problem. In the present work, we have formulated an analytical and separable representation of the two-nucleon S wave t matrix, both for the triplet and the singlet states for the local Yukawa potentials. We have also included the repulsive interaction through a

soft-core repulsive term in the singlet interaction. The S wave sturmian eigenfunctions for the attractive Hulthén potential for both the triplet and the singlet states which are known analytically (Sitenko *et al* 1968) have been used as the corresponding basis vectors for our expansion of the t matrix. This seems to be justified on the ground that the Yukawa and Hulthén potentials closely resemble each other in their nature. The resulting t matrices for the triplet and the singlet states, which have been expressed as the expansions over the corresponding sturmian sets of eigenfunctions in the form of double summations, have been found to maintain term-by-term symmetry in momentum variables p and p' thus obeying time reversal invariance explicitly. The coefficients in the summation for the expansion of the t matrix have been evaluated explicitly by solving a system of simultaneous linear equations connecting the coefficients. For values of $p, p' \leqslant 3\ \mathrm{fm}^{-1}$ and $(-MS)^{1/2} \leqslant 3\ \mathrm{fm}^{-1}$ (M being the nucleon mass and S being the energy), which are of importance in the calculation of the trinucleon binding energy (Levinger *et al* 1969), we have calculated the contributions of one and four terms in the expansion of the t matrix for both the triplet and the singlet states to study the rate of convergence of the separable expansion. A fairly good convergence is observed with respect to one term and four terms in the expansion. These t matrices have been used as the inputs in the Faddeev equation for the neutron–deuteron collision problem and the triton binding energy E_T has been computed by the numerical solution of the coupled integral equations which become one dimensional due to the separability of the t matrix in p and p'. Calculations for E_T have been performed with one term and four terms in the expansion of the t matrix for both the triplet and the triplet plus the singlet states. The results have been compared with those of Malfliet and Tjon (1969) as well as the experimental findings. The reasonably good convergence in the binding energy results with one term and four term expansions for the t matrix in the triplet interaction and the fairly good agreement of our results with the four term expansion for the triplet plus the singlet states with those of Malfliet and Tjon (1969) indicates the reliability of our separable expansion method for the t matrix corresponding to Yukawa interactions. Our method gives an analytical prescription for the separable representation of the off-shell partial-wave t matrix for Yukawa potentials. As is well known, the separability which exists in our formulation will enormously simplify the calculation by reducing the Faddeev equations to one dimensional integral equations.

2. Theory

The Lippmann–Schwinger integral equation for the two-body partial-wave t matrix is

$$t_l(p, p'; s) = V_l(p, p') - 4\pi \int_0^\infty \frac{V_l(p, q) t_l(q, p'; s)}{q^2/M - s} q^2\, \mathrm{d}q \tag{1a}$$

where

$$V(\boldsymbol{p}, \boldsymbol{p}') = \frac{1}{(2\pi)^{3/2}} \int V(r) \exp\{\mathrm{i}(\boldsymbol{p} - \boldsymbol{p}') \cdot \boldsymbol{r}\}\, \mathrm{d}\boldsymbol{r} \tag{1b}$$

$$V_l(p, p') = \tfrac{1}{2} \int_{-1}^{+1} V(\boldsymbol{p}, \boldsymbol{p}') \mathrm{P}_l(\cos\theta)\, \mathrm{d}\cos\theta \tag{1c}$$

where $V(r)$ stands for the interaction potential and θ is the angle between the vectors $\boldsymbol{p}$ and $\boldsymbol{p}'$ and all other symbols have their usual meanings. The inclusion of the soft-core

repulsion in the singlet state, only adds extra terms to $V(r)$. In our investigations, we have considered the attractive Yukawa potential in both the triplet and the singlet states and have included a soft-core repulsion in the latter as in Malfliet and Tjon (1969). The S state sturmian eigenfunctions for attractive Hulthén potentials in the triplet and the single states of the two-nucleon system, which are obtained analytically (Sitenko *et al* 1968) have been used as the corresponding basis vectors of our expansion for the t matrix for the triplet and the singlet states. The S state sturmian eigenfunctions $g_n(p, s)$ are solutions of the equations

$$\int_0^\infty \frac{V(p, q)g_n(q, s)q^2\,\mathrm{d}q}{q^2/M - s} = \lambda_n(s)g_n(p, s) \tag{2a}$$

with the usual orthonormal property

$$4\pi \int_0^\infty \frac{g_m(q, s)g_n(q, s)}{q^2/M - s} q^2\,\mathrm{d}q = \delta_{nm} \tag{2b}$$

where λ_n is the eigenvalue corresponding to the eigenfunction $g_n(p, s)$.

Hence the off-shell S wave t matrix can be expanded in a complete orthonormal set of sturmian eigenfunctions for the Hulthén potential, in the form of a double summation, namely

$$t(p, p'; s) = \sum_n \sum_m C_{mn}(s)g_n(p', s)g_m(p, s) = \sum_n \sum_m t_{mn}(p, p'; s) \tag{3}$$

where $C_{mn}(s)$ are the unknown expansion coefficients. Substituting (3) in (1a) and using the relation (2b), we get as many number of linear equations as there are unknown expansion coefficients. These equations can be solved and the coefficients $C_{mn}(s)$ can be explicitly evaluated. Our separable expansion with one term namely, $m = n = 1$, is

$$t_{11}(p, p'; s) = C_{11}(s)g_1(p, s)g_1(p', s) \tag{4a}$$

and with the four term expansion m and n assume values 1 and 2 and hence

$$\begin{aligned} t(p, p'; s) = {} & C_{11}(s)g_1(p, s)g_1(p', s) + C_{12}(s)g_1(p, s)g_2(p', s) + C_{21}(s)g_2(p, s)g_1(p', s) \\ & + C_{22}(s)g_2(p, s)g_2(p', s) \\ = {} & t_{11}(p, p'; s) + t_{12}(p, p'; s) + t_{21}(p, p'; s) + t_{22}(p, p'; s). \end{aligned} \tag{4b}$$

Substituting (4b) on both sides of (1a) and using the relation (2b), we get a set of four linear algebraic equations connecting the four coefficients. They can be solved explicitly and the off-shell t matrix is then known analytically. Similar is the case with (4a). Thus the t matrix is explicitly known in analytical and separable form. Since the exact t matrix $t(p, p'; s)$ possesses time reversal invariance, we have

$$t(p, p'; s) = t(p', p; s)$$

or

$$\begin{aligned} & C_{11}(s)g_1(p, s)g_1(p', s) + \{C_{12}(s)g_1(p, s)g_2(p', s) + C_{21}(s)g_1(p', s)g_2(p, s)\} \\ & \quad + C_{22}(s)g_2(p, s)g_2(p', s) + \ldots \\ & = C_{11}(s)g_1(p', s)g_1(p, s) + \{C_{12}(s)g_1(p', s)g_2(p, s) + C_{21}(s)g_1(p, s)g_2(p', s)\} \\ & \quad + C_{22}(s)g_2(p, s)g_2(p', s) + \ldots. \end{aligned}$$

Multiplying both sides by $g_1(p, s)g_2(p, s)(p^2/M - s)^{-1}(p'^2/M - s)^{-1}$ and integrating over

$d\boldsymbol{p}$ and $d\boldsymbol{p}'$ we get with the help of (2b), $C_{12}(s) = C_{21}(s) = C(s)$. Similarly, it can be proved that $C_{mn}(s) = C_{nm}(s)$. Hence our separable expansion with four terms is

$$t(p, p'; s) = C_{11}(s)g_1(p, s)g_1(p', s) + C(s)\{g_1(p, s)g_2(p', s) + g_1(p', s)g_2(p, s)\} + C_{22}(s)g_2(p, s)g_2(p', s) \ldots . \tag{4c}$$

Therefore, our separable expansion in the form (4c) maintains term-by-term symmetry in p and p', thereby obeying explicitly the principle of detailed balancing. To include the soft-core repulsion in the singlet state, we have added one extra repulsive term to the corresponding $V(r)$. This however, does not increase the number of separable terms in (4a) or (4b).

The use of the separable representation for the two-particle t matrix allows the three-particle Faddeev equations to be reduced to a set of one dimensional coupled integral equations. For the triton problem, which has spin $\frac{1}{2}$ and isotopic spin $\frac{1}{2}$, we arrive at the following set of homogeneous coupled equations (Schick and Hetherington 1967):

$$X_{ij}(S) = \sum_k \sum_l Z_{ik}(S)\mathscr{T}_{kl}(S)X_{lj}(S) \tag{5}$$

where

$$\langle q|\mathscr{T}_{kl}(S)|q'\rangle = \delta(q-q')T_{kl}(S-q^2/M)$$

and

$$\langle k|G_0(S)|i\rangle = Z_{ik}(S)$$

$G_0(s)$ is the free three-particle Green function and all other symbols have their usual meanings. Here

$$t_{kl}(p, p'; s) = \tilde{F}(p)T_{kl}(s)F(p')$$

with

$$\tilde{F}(p) = (g_1(p), g_2(p), \ldots)$$

being the transpose of the column vector $F(p)$ and the matrix $T(s)$ is given by

$$T(s) = \begin{pmatrix} C_{11} & C_{12} & \cdots \\ C_{12} & C_{22} & \cdots \\ \vdots & & \ddots \end{pmatrix}.$$

We have carried out calculations for E_T with (4a) and (4c) as inputs in (5) for both the triplet and the triplet plus the singlet states.

3. Results and discussions

For our calculations, we have used the same two-body input data as used by Malfliet and Tjon (1969) in their triplet IV and singlet I potential. We have used for the nucleon–nucleon interaction in the ^{1}S channel a superposition of two Yukawa potentials one of which is repulsive, that is $V(r) = -\lambda_A(e^{-\mu_A r}/r) + \lambda_R(e^{-\mu_R r}/r)$. The potential constants have been determined from a fit to the low energy parameters and phase shifts up to

300 MeV laboratory energy (Malfliet and Tjon 1969). We have $\lambda_A = 0{\cdot}330$, $\mu_A(\text{fm}^{-1}) = 0{\cdot}633$ and $\lambda_R = 0$ in the ^{3}S state and $\lambda_A = 2{\cdot}64$, $\mu_A(\text{fm}^{-1}) = 1{\cdot}55$, $\lambda_R = 7{\cdot}39$, $\mu_R(\text{fm}^{-1}) = 3{\cdot}11$ for the ^{1}S state. The triplet parameters yield the scattering length $a = 5{\cdot}45$ fm, binding energy of deuteron equal to 2·23 MeV and effective range $r = 1{\cdot}8$ fm. The singlet parameters yield $a = -23{\cdot}3$ fm and $r = 2{\cdot}8$ fm.

The sturmian eigenfunctions corresponding to the (attractive) Hulthén potential for both the triplet and the singlet states have been chosen as the basis vectors of our separable expansion for the t matrix. The numerical values of the parameters used in the sturmian eigenfunctions corresponding to the Hulthén potentials are the same as those used by Kharchenko and Petrov (1969) in their investigation of the neutron–deuteron collision problem with the Hulthén potential. Thus with $V(r) = -V_0(e^{kr}-1)^{-1}$ where $v_0 = 1{\cdot}8509\ \text{fm}^{-2}$, $k = 1{\cdot}148\ \text{fm}^{-1}$ for ^{3}S state and $v_0 = 1{\cdot}3493\ \text{fm}^{-2}$, $k = 1{\cdot}202\ \text{fm}^{-1}$ for ^{1}S state we get from (2*a*) (Sitenko *et al* 1968):

$$g_n(p, s) = A_n(s) \sum_{\nu=1}^{n} B_{n\nu}(s) v_0 \frac{1}{p^2 - Ms + (v_0/\lambda_\nu(s))}$$

where

$$v_0 = \frac{MV_0}{\hbar^2} \qquad \lambda_n(s) = v_0\{(nk - 2i\sqrt{(Ms)})nk\}^{-1}$$

$$A_n^{-2}(s) = 2\pi^2 Mk^2 \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} \mu\nu B_{n\mu}(s) B_{n\nu}(s) \frac{1}{(\mu+\nu)k - 2i\sqrt{(Ms)}}$$

$$B_{n1}(s) = \lambda_1(s) \qquad B_{n\nu}(s) = \frac{-n+\nu-1}{n+\nu-1} \frac{\lambda_\nu(s)}{\lambda_{n+\nu-1}(s)} B_{n+\nu-1}(s).$$

We have compared the t matrices with (4*a*) and (4*c*) for the triplet and the singlet states for p, $p' \leqslant 3{\cdot}0\ \text{fm}^{-1}$ and $\sqrt{(-MS)} \leqslant 3{\cdot}0\ \text{fm}^{-1}$ which are of importance in the calculation of the trinucleon binding energy and values have been listed in tables 1 and 2. The good agreement in the tabulated results for $t(p, p'; s)$ with (4*a*) and (4*c*) for the above ranges of values of p, p' and s, indicates reasonably good convergence of the separable expansion for both the triplet and the singlet states.

With the inputs (4*a*) and (4*c*) for t matrices in the Faddeev equations (5), we arrive at the set of one dimensional homogeneous integral equations. Considering only the triplet state with (4*a*) and (4*c*) as inputs, we get one and two coupled equations respectively. With the inclusion of the triplet and the singlet states (with soft-core repulsion) in (5), we arrive at two or four coupled integral equations according as (4*a*) or (4*c*) is used

Table 1. The t matrices for Yukawa interactions in the triplet state have been tabulated for the one term expansion $t_1(p, p'; s)$ and the four term expansion $t_4(p, p'; s)$

$\sqrt{(-MS)}$ (fm^{-1})	t_1			t_4		
	$t(1,1;s)$	$t(1,2;s)$	$t(1,3;s)$	$t(1,1;s)$	$t(1,2;s)$	$t(1,3;s)$
0·4	−0·3568	−0·3234	−0·2939	−0·3736	−0·3355	−0·2518
0·6	−0·2135	−0·1877	−0·1634	−0·2345	−0·2031	−0·1432
0·9	−0·082	−0·0655	−0·0413	−0·1250	−0·0800	−0·0387
2	−0·016	−0·0143	−0·0106	−0·0287	−0·0022	−0·0145
3	−0·0076	−0·0073	−0·0059	−0·0110	−0·0120	−0·0080

Table 2. The t matrices for the Yukawa interaction in the singlet state (with soft-core repulsion) have been tabulated for the one term expansion $t_1(p, p'; s)$ and the four term expansion $t_4(p, p'; s)$

$\sqrt{(-MS)}$ (fm^{-1})	t_1			t_4		
	$t(1,1;s)$	$t(1,2;s)$	$t(1,3;s)$	$t(1,1;s)$	$t(1,2;s)$	$t(1,3;s)$
0·4	−0·6770	−0·3808	−0·2708	−0·7866	−0·3859	−0·2921
0·6	−0·5947	−0·3483	−0·2314	−0·6809	−0·3376	−0·2500
0·9	−0·8146	−0·5351	−0·0413	−0·9068	−0·4698	−0·0387
2	−0·7468	−0·5944	−0·0106	−0·8310	−0·4769	−0·0145
3	−0·2710	−0·2347	−0·1918	−0·2933	−0·1831	−0·1674

as input for the t matrix for both the triplet and the singlet states. The integration variable in (5) is then transformed suitably so as to change the limits of integration into 1 and −1. The set of coupled homogeneous integral equations is then recast into a matrix equation with the help of the Gauss–Legendre quadrature method. The triton binding energy E_T is then computed by searching for the pole in the inverted matrix. This is located by finding out the zero of the corresponding determinant. We have used the 20-point Gauss–Legendre quadrature for our calculations for E_T. To check the convergence and hence the accuracy of the results, we have made further calculations with 24 quadrature points and it is found that the results agree very reasonably. We have carried out calculations with one term (4*a*) and four terms (4*b*) with the triplet and the triplet plus the singlet state (with soft-core repulsion). With the triplet state only, our results for E_T with one term (4*a*) and four terms (4*b*) are in close agreement with each other indicating good convergence. Considering both the triplet and the singlet states we obtain, with one term expansion (4*a*) for the t matrix, the binding energy value as 9·91 MeV, which is higher than Tjon's result 8·4 MeV. When the four-term expansion for the t matrix (4*c*) is used we obtain the result 9·20 MeV, which is in closer agreement with Tjon's result than before. To study the convergence of the result more terms in the separable expansion of the t matrix should be used. This will, however, lead to enormous computational difficulty as the number of coupled integral equations will become prohibitively large. In table 3, we have tabulated the results of the present calculation for the values of the triton binding energies together with the theoretical result of Malfliet and Tjon (1969) and the experimental findings. A comparison of the results

Table 3. The triton binding energy values E_T (in MeV)

Present calculations				Malfliet and Tjon with triplet and singlet states (with soft-core repulsion)	Experimental result
with triplet state		with triplet plus singlet states (with soft-core repulsion)			
one term	four terms	one term	four terms		
6·65	6·86	9·91	9·20	8·4	8·5

indicates that our prescribed method for the construction of an analytical and separable form of the S wave t matrix for Yukawa interactions yields reasonably good results in the three-body binding energy calculation.

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